The Annual Meeting of the Population Association of America,
Dallas, TX, April 15-17, 2010

Multistage Models of First Marriage and Birth: An Extension of the Coale-McNeil Nuputiality Model[1]

Ryuichi Kaneko[2]

Abstract

In this study, I first focus on the potential profile of the Coale-McNeil nuptiality model (CM model) as a multistage model of vital events such as the first marriage and birth separated by birth order. The convolution structure of the CM model can be regarded as an expression of a multistage process consisting of attainment of marriageable age, and several waiting time to the goal. This interpretable nature of the CM model is carefully examined here and the corresponding equivalent convolution model is presented. The critical problem is that the assumption of the model over independence between the sub-processes is not satisfied in the reality according to my survey results. Hence I proposed new model by introducing liner relationships among parameters so that realistic associations of the sub-processes be reproduced. It is applicable to describe latent processes such as age at becoming marriageable and marriage market population, for instance.

Introduction

The marriage model developed by Coale and McNeil (1972) provides an elegant example of a multistage model that might be applied to a wide range of events. Their model is a limiting probability distribution of the convolution of infinite numbers of related exponential distributions, and can be regarded as the convolution of a uni-modal distribution of its own form and some number of exponential distributions. Therefore it provides a general mathematical model applicable to multistage processes, by which I mean a process that consists of multiple processes whose completion is required for the target event to occur[3]. Unfortunately, this aspect of the CM model has drawn only limited attention in contrast to its popularity as a standard schedule of first marriage. In this paper, I focus on this profile of the CM model, and examine its potential ability as a behavioral

---

[1] This is an early version of the study report, or an extended abstract of the paper to be presented to The Annual Meeting of the Population Association of America (Dallas, TX, April 15-17, 2010).

[2] National Institute of Population and Social Security Research, Japan. e-male: r-kaneko@ipss.go.jp

[3] The multistage model in this context is a special case of the multistate model in which return transitions to previous states are prohibited.

multistage model of the first marriage process.

Multistage Framework of First Marriage Process

Coale and McNeil viewed first marriage as a multistage process in which entry into marriageable state, meeting of the eventual spouse, and engagement are required to take place prior to the marriage[4] (Coale and McNeil 1972). They proposed convolution models to formalize this view of the process. Their model for the first marriage schedule is viewed here as a behavioral model of the process as well, since it is composed of a convolution of multiple distributions which can individually be interpreted behaviorally. Coale and McNeil also proposed a model consisting of a convolution of a normal and some exponential distributions solely for extracting a behavioral explanation. Before examining these models, I first formulate a framework in which to study the multistage aspect of the first marriage process.

Let the following random variables be components of the first marriage process;

Z : age at first marriage
X_0: age at entry into marriage market (age at becoming marriageable)
X_1: age at first encounter with eventual spouse
X_2: age at engagement
T_1: waiting time from entry into marriage market to encounter
T_2: waiting time from encounter to engagement
T_3: engagement period

Then the following relationships hold among these variables in a first marriage process:

$$\begin{aligned} Z &= X_0 + T_1 + T_2 + T_3 \\ &= X_1 + T_2 + T_3 \\ &= X_2 + T_3 \end{aligned} \tag{1}$$

According to this formulation, the process starts with entry into marriage market and goes through three waiting periods by the time of marriage. Thus the age at first marriage is defined as the end point of the process. The second and third lines of the equation indicate that the beginning of the process may be age at first encounter with spouse, or age at engagement.

Entry into the marriage market is merely a time point at which a person becomes eligible to marry. Physical maturity, arrival at the legal age for marriage, satisfaction of social and economic requirements such as completion of school or entering employment, and psychological readiness for marriage are all considered as necessary conditions of the

[4] They acknowledged Griffith Feeney for initial inspiration of this view.

entry. Since the specific events that cause a person to enter the marriage market vary widely from person to person and may well be largely psychological, this notion may be best viewed as an abstraction to assist in description of the model.

According to Coale and McNeil (1972), X_1 should be the time of "meeting or stating to keep frequent company with or beginning to date the eventual" spouse. However, I employ "meeting with eventual spouses" as the event for X_1 due to ease of observation[5].

In order to develop a specific demographic model out of this basic framework, I should identify probability distributions for each variable and the relationships among them. The first attempt to assemble a multistage model should be conducted via the convolution of distributions that are allocated to sub-processes.

The convolution provides the distribution of the sum of two random variables, assuming they are mutually independent. Let X and T denote two independent random variables, which have PDFs, $f(x)$ and $h(t)$ respectively. Then the PDF of Z=X+T, $g(z)$ is given by[6]:

$$g(z) = \int_{-\infty}^{\infty} f(x)h(z-x)dx\,. \tag{2}$$

For instance, suppose that X is age at entry into marriage market and T is waiting time until marriage, at which point age is Z. Then distribution of age at marriage is given by $g(z)$.

Some convolutions of standard distributions are well known. For instance, the convolution of two normal distributions is another normal distribution with sum of the original means as the new mean, and sum of the original variances as the new variance. The convolution of multiple identical exponential distributions is a gamma distribution with two parameters, one of which is the number of exponentials, and the other is the parameter of the exponential distributions. I should note that the independence of the random variables is required when applying convolution.

The framework described here is applicable to many compound events. Birth is one example in which several events are required to take place prior to the event, such as marriage or union formation, onset or resumption of ovulation, copulation, and conception. Some authors have employed multistage framework with convolution models to analyze birth interval (D' Souza 1974, Zhu 1994, and Liang 2000).

The Coale-McNeil Distribution as a Convolution Model

Coale and McNeil (1972) proposed two different types of convolution models for the

[5] A drawback of this specification is that meetings in childhood in such a case as marriage between neighbors are included. Those meetings occur much earlier than entry into the marriageable state. But according to our survey described later, such meetings are exceptional in present-day Japan (1.5% in '97 survey). Further, it is possible to eliminate them based on information regarding the type of meeting.

[6] If T>0 holds, then the upper boundary of the integral should be z instead of plus infinity. This is the case when T stands for waiting time.

first marriage process; (1) a distribution that is a limiting distribution from the convolution of infinite numbers of related exponentials with means in a harmonic sequence, and (2) a distribution given by the convolution of the normal distribution and several exponential distributions. First I examine the form and properties of model (1) here. Model (2) is discussed in the following section.

In terms of probability density function (PDF), the model (1) is given by;

$$g(x) = \frac{\beta}{\Gamma(\alpha/\beta)} \exp\left[-\alpha(x-\mu) - \exp\{-\beta(x-\mu)\}\right]$$

where Γ denotes the gamma function[7], $\alpha(>0)$, $\beta(>0)$, and $\mu(-\infty<\mu<\infty)$ are three parameters (Coale and McNeil 1972). More precisely this was derived as the PDF of a random variable X such that

$$X = \left(\mu - \frac{\psi(\alpha/\beta)}{\beta}\right) + \sum_{j=1}^{\infty}\left(T_j - \frac{1}{\alpha+(j-1)\beta}\right)$$

where ψ denotes the digamma function, and T_j is a random variable that follows the exponential distribution with mean $1/\{\alpha+(j-1)\beta\}$ (Coale and McNeil, 1972).

Since model (1) given above is made up of infinite numbers of related exponential distributions, it can be viewed as a convolution of itself and some number of exponential distributions. According to equation of *X* above, the exponential distribution with the largest mean (smallest hazard) in the CM distribution has the parameter (hazard) α. Elimination of this distribution from the CM distribution makes another CM distribution with α substituted by $\alpha+\beta$. In general removal of the exponential distributions with the *m*-th largest means makes another CM distribution with α substituted by $\alpha+m\beta$. Specifically, letting $h_T(t;m)$ denote the PDF of the convolution of the exponential distributions with the *m*-th largest means and $g_X(x;m)$ be the PDF of the CM distribution resulting from removal of $h_T(x;m)$, those PDFs are given by:

$$g_X(x;m) = \frac{\beta}{\Gamma(\alpha/\beta+m)} \exp\left[-(\alpha+m\beta)(x-\mu) - \exp\{-\beta(x-\mu)\}\right] \quad (3)$$

$$h_T(t;m) = \frac{\beta\Gamma(\alpha/\beta+m)}{\Gamma(\alpha/\beta)(m-1)!}\{1-\exp(-\beta t)\}^{m-1}\exp(-\alpha t) \quad (4)$$

where α, β, and μ are three parameters of the CM distribution (Coale and McNeil, 1972).

Here $g_X(x;m)$ represents a distribution of age at entering a stage from which the

---

[7] The gamma function is here defined as:

$$\Gamma(y) = \int_0^{\infty} u^{y-1}e^{-u}du.$$

process starts, and $h_T(t;m)$ is the distribution of the waiting time that is composed of m exponentially distributed waiting times. The mean and variance of the distribution $g_X(x;m)$ are respectively $\mu - \frac{1}{\beta}\psi\left(\frac{\alpha}{\beta}+m\right)$, and $\frac{1}{\beta^2}\psi'\left(\frac{\alpha}{\beta}+m\right)$.

Those of the distribution $h_T(x;m)$ are respectively $\sum_{j=1}^{m}\{\alpha+(m-1)\beta\}^{-1}$, and $\sum_{j=1}^{m}\{\alpha+(m-1)\beta\}^{-2}$.

As mentioned above, the exponential distribution with the largest mean convolved in distribution $h_T(t;m)$ has the parameter α. This is supposedly a distribution of the duration from entry into the marriage market to the meeting of the future husband. The exponential distributions of the second and third largest mean have the parameters, $\alpha+\beta$, and $\alpha+2\beta$. These are supposed to be distributions of the durations of dating and engagement. According to the parameter values of the CM standard age distribution of first marriage, which are derived from experiences of Swedish female cohorts, the mean duration from entry into marriage market to meeting of future husband is estimated as ($1/0.174$) or 5.75 years. Similarly means of the second and third waiting durations are 2.16 years ($1/(0.174+0.2881)$) and 1.33 years ($1/(0.174+2\times 0.2881)$) respectively (Coale and McNeil 1972).

As shown above, estimated parameter values for the CM distribution provides interesting behavioral measures as well. These interpretable features of the CM distribution are another advantage of the model in addition to its descriptive function.

Since the CM distribution is an alternative form of the generalized log-gamma distribution (GLG distribution) as identified by Kaneko(2003), the convolution formulations (5) and (6) are expressed in terms of the latter distribution. Namely:

$$g_X(x;m) = \frac{|\lambda|}{b\Gamma(\lambda^{-2}+m)}(\lambda^{-2})^{\lambda^{-2}+m}\exp\left[\left(\lambda^{-2}+m\right)\lambda\left(\frac{x-u}{b}\right)-\lambda^{-2}\exp\left\{\lambda\left(\frac{x-u}{b}\right)\right\}\right] \tag{5}$$

$$h_T(t;m) = \frac{|\lambda|\Gamma(\lambda^{-2}+m)}{b\Gamma(\lambda^{-2})(m-1)!}\left\{1-\exp\left(\frac{\lambda t}{b}\right)\right\}^{m-1}\exp\left(\frac{t}{b\lambda}\right) \tag{6}$$

where $\lambda\ (-\infty<\lambda<\infty, \neq 0), u\ (-\infty<u<\infty), b\ (>0)$ are three parameters, and Γ denotes the gamma function.

Some Other Convolution Models for First Marriage Process

Coale and McNeil also suggested a second type of convolution model for the first marriage process. In the course of deriving the first marriage model of the first type, they realized that repeatedly removing the exponential distribution from the CM distribution produced residual distributions that increasingly resemble the normal distribution. As a matter of fact, with random variable Z that follows the CM distribution, and T_j that follows the exponential distribution with mean $1/\{\alpha+(j-1)\beta\}$, they proved that $\sqrt{m}\left\{Z-\sum_{j=1}^{m}T_j-\left(a-\sum_{j=1}^{m}\frac{1}{\alpha+(j-1)\beta}\right)\right\}$ has a limiting distribution as $m\rightarrow\infty$ which is the normal distribution with mean zero and variance $1/\beta^2$ (Coale and McNeil 1972).

This implies that the CM distribution can be approximated by convolution of the normal distribution and some of the exponential distributions. After examining the cumulants of the distributions, they concluded that when m=3 the convolution is sufficiently close to their standard model derived from Swedish experience.

They did not derive any mathematical form of the convolution with a PDF or CDF in closed form. However, this formulation has certain technical advantages because of the abundance of techniques for working with the normal distribution. The basic formula for the convolution of the normal distribution and some distribution of non-negative waiting time with PDF $w(t)$ is given by:

$$g(z)=\int_{-\infty}^{z}\frac{1}{\sigma_e}\phi(\frac{t-\mu_e}{\sigma_e})w(z-t)dt \tag{7}$$

where $\phi(x)$ is PDF of the standard normal distribution.

In the following, $N(\mu,\sigma)$ denotes the normal distribution with mean μ and standard deviation σ, and $E(\gamma)$ denotes the exponential distribution with mean $1/\gamma$. A$\oplus$B stands for convolution of distributions, A and B. $\phi(x)$, $\Phi(x)$ are respectively the PDF and CDF of the standard normal distribution.

The convolution of the normal and one exponential distribution $N(\mu_e,\sigma_e)\oplus E(\gamma)$ is formulated by D'Souza (1974). Its PDF and CDF are given by:

$$g(z)=\gamma h(z) \tag{8}$$

$$G(z)=\Phi(\frac{z-\mu_e}{\sigma_e})-h(z) \tag{9}$$

where $h(z)=\exp\left\{-\gamma(z-\mu_e)+\frac{1}{2}\sigma_e^2\gamma^2\right\}\Phi(z^*)$, in which |.

Supposing that the normal distribution controls age at entry into marriage market, and the exponential distribution controls waiting time to marriage, I can derive the proportion who are in the marriage market at age x, $P_E(x)$, and the hazard rate of marrying for those in the market, $r_E(x)$, as:

$$P_E(x) = h(x)$$
$$r_E(x) = \gamma$$

Following the similar procedure, I formulate the convolution of the normal and two different exponential distributions (, $\gamma_1 \neq \gamma_2$). Its PDF and CDF are given by:

$$g(z) = \frac{\gamma_1\gamma_2}{\gamma_2 - \gamma_1}\{h_1(z) - h_2(z)\} \tag{10}$$
$$G(z) = \Phi(\frac{z-\mu_e}{\sigma_e}) - \frac{\gamma_1\gamma_2}{\gamma_2-\gamma_1}\left\{\frac{1}{\gamma_1}h_1(z) - \frac{1}{\gamma_2}h_2(z)\right\} \tag{11}$$

where $h_i(z) = \exp\left\{-\gamma_i(z-\mu_e) + \frac{1}{2}\sigma_e^2\gamma_i^2\right\}\Phi(z^*)$, in which $|$, $i = 1,2$.

The proportion who is in the marriage market at age *x*, and the hazard rate of marrying for those in the market are given by:

$$P_E(x) = \frac{\gamma_1\gamma_2}{\gamma_2-\gamma_1}\left\{\frac{1}{\gamma_1}h_1(z) - \frac{1}{\gamma_2}h_2(z)\right\}$$
$$r_E(x) = \frac{h_1(x) - h_2(x)}{h_1(x)/\gamma_1 - h_2(x)/\gamma_2}$$

To compute the convolution of the normal and two identical exponential distributions, observe that by the associativity of convolution the exponentials convolve to a gamma distribution. Thus I can view the total convolution as the convolution of the normal and gamma distributions. Its PDF and CDF are given by

$$g(z) = \sigma_e\gamma^2\left\{z^*h(z) + \phi(z^*)\right\} \tag{12}$$
$$G(z) = \Phi(\frac{z-\mu_e}{\sigma_e}) - \left\{h(z) + \frac{1}{\gamma}g(z)\right\} \tag{13}$$

where $h(z) = \exp\left\{-\gamma(z-\mu_e) + \frac{1}{2}\sigma_e^2\gamma^2\right\}\Phi(z^*)$.

The proportion who is in the marriage market at age *x*, and the hazard rate of marrying for those in the market are given by:

$$P_E(x) = h(x) + \frac{1}{\gamma}g(x)$$
$$r_E(x) = \frac{h(x)}{h(x) + g(x)/\gamma}$$

Non-independent "Convolution" Model

All those formulae discussed above depend critically on the assumption of independence between the distributions to be convolved. If this assumption does not hold for the actual target process, their utility as behavioral models is limited. This issue has not been discussed extensively in previous work due to a lack of empirical evidence.

Kaneko (1991a) pointed out the heavy dependence of waiting time (T2+T3) on age at first meeting (X1) based on survey data collected in Japan[8]. He reported, in particular, a strong negative correlation between X1 (age at first encounter with eventual spouse) and T2 (waiting time from encounter to engagement) as -0.449 for Japanese women with sample size N=4682. These correlations make variance of age at first marriage much smaller than that of age at first meeting, which should not happen in the convolution framework. Hence the validity of convolution models of first marriage process as behavioral explanatory tools is questionable at least for some populations.

Therefore, in order to develop the behavioral multistage models I should consider models without convolution or with non-independent "convolution", by which I mean a procedure generating the distribution of a sum of random variables assuming dependencies among them. The general formula for the PDF, $g_z(z)$, of sum of two random variables X and T (>0) is given by:

$$g_z(z) = \int_{-\infty}^{z} g_x(x) w_t(z-x|x)dx \tag{14}$$

where $g_x(x)$ is PDF of X and $w_t(t|x)$ is conditional PDF of T given X=x.

With this integral formula, I would obtain $g_z(z)$ by providing functional form of $g_x(x)$ and $w_t(t|x)$.

One possible such model is one whose parameters of the waiting time distributions are functions of age at initiation of waiting, i.e. X. Here I attempt to develop a model of this type to conduct a numerical examination, in a search for appropriate models to describe the observed of multistage process.

I employ the generalized gamma (GG) distribution to represent the mathematical models for waiting time because it is one of the most flexible parametric models for survival times. Its PDF is given by:

$$w(t;\theta,c,v) = \frac{|\theta|}{c\Gamma(\theta^{-2})}(\theta^{-2})^{\theta^{-2}}\left(e^{v}t\right)^{\frac{1}{c\theta}-1}\exp\left[-v-\theta^{-2}\left(e^{v}t\right)^{\frac{\theta}{c}}\right] \tag{15}$$

[8] The Ninth National Fertility Survey (NFS 9) conducted in 1987 in Japan by Institute of Population Problem (currently National Institute of Population and Social Security Research).

where $\theta, c,$ and v are three parameters.

Let the scale parameter v be dependent on age at onset x, and expressed as a polynomial function of x, such as $v(x) = v_0 + v_1 x + v_2 x^2 \cdots$, where $v_0, v_1, v_2 \cdots$ are polynomial coefficients. Then PDF of the GG waiting time distribution is given by

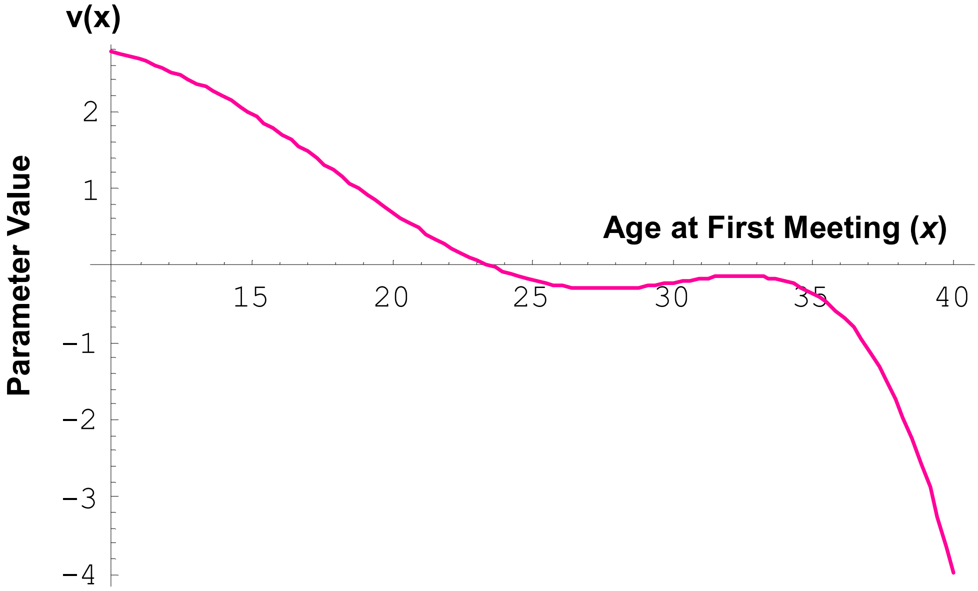
$$w(t, x; \theta, c, v_0, v_1, v_2 \cdots) = \frac{|\theta|}{c\Gamma(\theta^{-2})} (\theta^{-2})^{\theta^{-2}} \left(e^{v(x)} t\right)^{\frac{1}{c\theta}-1} \exp\left[-v(x) - \theta^{-2} \left(e^{v(x)} t\right)^{\frac{\theta}{c}} \right]. \tag{16}$$

It is possible to make the other parameters, θ and c dependent on X in the same way, though I do not attempt it here.

Parameter estimation of $w(t, x; \theta, c, v_0, v_1, v_2 \cdots)$ is conducted by applying it to survey data of delay from age at first meeting to marriage in Japanese women through the maximum likelihood method[9]. The result indicates that the polynomial functions up to five produces statistically significant improvements in fitness, but little improvement is seen thereafter. Function $v(x)$, the scale parameter value of the GG waiting time for those whose age at first meeting is x, is shown in Figure-1 in the form of a fifth degree polynomial.

[9] The Ninth through Eleventh National Fertility Survey conducted in 1987, 1992, and 1997 in Japan by National Institute of Population and Social Security Research.

Figure 1 Scale Parameter of the GG Distribution for Waiting Time as a Function of Age at First Meeting with Eventual Husband: Japanese Women

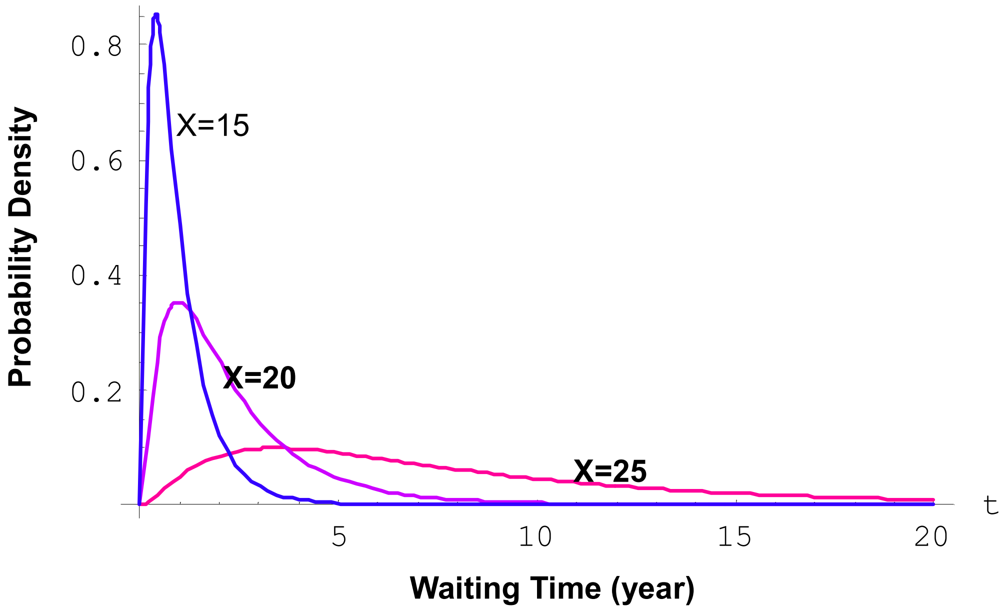


Note: Estimated fifth degree polynomial as a parameter of the GG distribution for waiting time from age at first meeting with eventual husband to marriage. Source data is from The Ninth through Eleventh National Fertility Survey conducted in 1987, 1992, and 1997 in Japan by National Institute of Population and Social Security Research.

As age at meeting increases, the value of v goes down until mid twenties. Then it stabilizes at slightly below zero until the mid thirties followed by a steep fall. Note that the rapid decline after the late thirties is not reliable due to the small sample size.

Corresponding waiting time distributions for ages 15, 20, and 25 at first meeting are shown in Figure-2. The figure indicates that there are remarkable differences in waiting time distribution depending on age at first meeting, which implies again that convolution model with the independence assumption is simply not realistic.

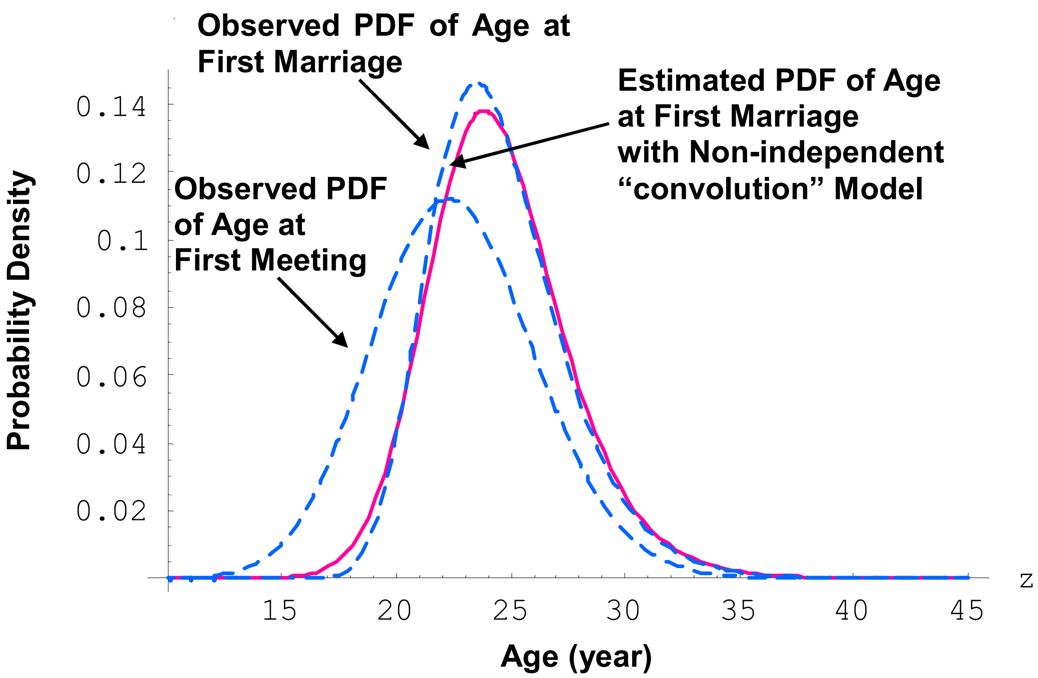
Figure 2 PDF of Estimated GG Distribution for Waiting Time by Age at First Meeting with Eventual Husband: Japanese Women



Note: The estimated the GG distribution for waiting time from age at first meeting with eventual husband to marriage. Source data is from The Ninth through Eleventh National Fertility Survey conducted in 1987, 1992, and 1997 in Japan by National Institute of Population and Social Security Research.

With this waiting time distribution combined with the estimated GLG distribution of age at first meeting on the same sample, age distribution of first marriage is reconstructed by numerical integration of the non-independent convolution formula (14) and is compared with directly estimated GLG distribution. The result is shown in Figure-3.

Figure 3 Observed and Estimated PDF of Age Distributions of First Marriage: Japanese Women



Note: Estimated PDF of age at first marriage with non-independent "convolution" model is compared with estimated PDF of the GLG model. Estimated PDF of age at first meeting with eventual husband is also shown. Source data is from The Ninth through Eleventh National Fertility Survey conducted in 1987, 1992, and 1997 in Japan by National Institute of Population and Social Security Research.

In the figure the reconstructed PDF of age at first marriage with the non-independent "convolution" of age at first meeting and dependent waiting time is matched up to the estimated PDF of the CM distribution of age at first marriage. The estimated PDF of age at first meeting with the eventual husband, which is used in the reconstruction, is also shown in the figure. The reconstructed distribution with non-independent convolution is fairly close to the directly estimated distribution, although differences are noticeable. Since the discrepancies remain even after introducing the dependence into the other parameters, it is due to inadequacy of the GG distribution for representing the waiting times, despite of the fact that the GG is one of the most flexible parametric models.

However, the examination indicates the possibility of employing the non-independent "convolution" model as a behavioral explanatory tool. It is shown that the model is able to reproduce distribution of age at first marriage fairly accurately, if the dependency of waiting time on age at meeting is properly represented.

Tentative Conclusion

In this paper, the CM model is enhanced as a behavioral model so as to describe latent processes behind occurrences of first marriage such as entering into marriage market and searching for a mate.

The CM model is viewed as a multistage model, by which we mean a process that consists of multiple processes required for a target event (*i.e.* marriage in this case) to occur. Its convolution structure can be regarded as an expression of a multistage process consisting of attainment of marriageable age, waiting time for meeting the eventual spouse, dating stage, and engagement period. This interpretable nature of the CM model is carefully examined and the corresponding equivalent convolution structure of the GLG model is presented. Various other convolution models for first marriage process are also examined. The critical problem associated with the multistage view of the CM model is that the assumption of the model over independence between the sub-processes is not satisfied in the reality. The analysis on a national representative survey of the first marriage process in Japan detected the substantial dependency between age at first meeting of a couple and subsequent waiting time for the first marriage (Kaneko 1991a, Kaneko 1999). This is likely the case for other societies. Hence we proposed a multistage model in which the independence among the sub-processes is not assumed (here, we call it non-independent "convolution" model) by introducing liner relationships among parameters so that realistic measurement of the sub-processes can be obtained. Though it may spoil the elegance to some extent, the model acquires additional applicability to the actual process. Our verification of the new model indicates reasonable agreements with the empirical data. It demonstrates the outstanding potential of the CM model to represent the behavioral process of first marriage and conceivably other demographic events as well such as birth.

The new model is to be applicable in many ways to behavioral studies of such vital events as marriage and birth separated by birth order. For example, it can be applied to estimation of size of marriage market, i.e. population size of marriageable people in a society. The size is possible to measure by estimating age distribution of people's becoming marriageable, which is major part of the model. The model should be effective to detect latent processes of births, since they are viewed as multistage processes. For application, it should be useful for estimating the distribution of "real" age at onset of sexual union by applying the model to the first birth for the society in which marriage is no longer viewed as the onset of sexual relationships among couples. Some of these applications should be demonstrated in the complete version of the present paper.

Selected References

Coale, A.J., McNeil, D. R., 1972, "The Distribution by Age of the Frequency of First Marriage in Female Cohort," *Journal of American Statistical Association*, Vol. 67: 743-749.

D'Souza, S., 1974, *Closed Birth Intervals: A Data Analytic Study*, New Delhi: Sterling Publishers.

Kaneko, R., 1991, “Demographic Analysis of First Marriage Process,” *The Journal of Population Problems*, Vol.47, No. 3: 3-27 (in Japanese with English summary)

—, 1999, “Below-replacement fertility in Japan: Trends, Determinants and Prospects,” 1999, 6, *Population Bulletin of the United Nations*, Nos. 40/41, pp.266-291

−−, 2003, “Elaboration of the Coale-McNeil Nuptiality Model as The Generalized Log Gamma Distribution: A New Identity and Empirical Enhancements,” *Demographic Research*, Vol. 9-10:223-262. http://www.demographic-research.org/Volumes/Vol9/10/9-10.pdf

Liang, Z. 2000. *The Coale-McNeil model. Theory, generalisation and application.* Amsterdam, The Netherlands: Thela Thesis.