COHORT POSTPONEMENT AND PERIOD MEASURES
EXTENDED ABSTRACT FOR PAA 2010

JOSHUA R. GOLDSTEIN AND THOMAS CASSIDY
MAX PLANCK INSTITUTE FOR DEMOGRAPHIC RESEARCH

ABSTRACT. We introduce a new class of models in which demographic behavior such as fertility is postponed by differing amounts depending only on cohort membership. We show how this model fits into a general framework of period and cohort postponement that includes the existing models in the literature, notably those of Bongaarts and Feeney and Kohler and Philipov. The cohort-based model shows the effects of cohort shifts on period fertility measures and provides an accompanying tempo-adjusted measure of period total fertility in the absence of observed shifts. Simulation reveals that when postponement is governed by cohorts, the cohort-based indicator outperforms the Bongaarts and Feeney model that is now in widespread use.

1. INTRODUCTION

One way to view much of the demographic change that is taking place in advanced societies is as a result of the changing meaning of age. One often hears that 40 is the new 30, or even 80 is the new 60. Demographers have developed formal models to show that shifting age-schedules (or equivalently, shifting meanings of age) can produce dramatic changes in cross-sectional period measures. Most notably, Bongaarts and Feeney's paper on fertility postponement, and their introduction of a "tempo-adjusted Total Fertility Rate" have become a fundamental part of the modern demographic toolkit.

The transformation of the human life cycle is a process that takes place within individual lives and is thus most naturally conceptualized as a cohort process. The magic of the Bongaarts and Feeney "tempo adjustment" is that only period data is needed. This is because postponement is modeled in period manner, with all ages (and thus all cohorts) postponing their events (or changing their clocks) in the same manner in a given year. This rate of change can change from period

Thanks to the MPDIR Tempo working group for comments on an earlier presentation of some of these results.

to period, but all cohorts must be treated the same. An enormous advantage of this approach – apart from any degree of realism it may or may not have – is that it produces a wonderfully simple mathematical model, in which changes in period mean ages completely determine the presence and extent of tempo effects.

Criticisms have been leveled at the Bongaarts and Feeney approach. Notably, a number of authors have stated that the uniform postponement across all ages is unrealistic (...). However, few authors have shown the consequences of departure from this assumption, or have even proposed alternatives. Zeng Yi and Kenneth Land have shown, using a set of simulations, that violations of the uniform postponement by age assumption matter relatively little. Kohler and Philipov proposed a major extension to the B&F, giving a fairly general framework for age and time varying postponement, and offering a special case in which postponement differed linearly by age within any period. The Kohler and Philipov results have not been widely used, in part because the paper is complex, but also because the estimation procedures needed are not thought to be particularly robust to random influences.

In this paper, we offer a conceptually simple alternative to the period-paramount view of postponement. Rather than uniform postponement by age within each period, we present a model in which there is uniform postponement in each cohort. Rodriguez (2006) provides inspiration by showing the relationship between these two views in the special case of linear shifts – that is when the rate of postponement (be it period or cohort) is unchanging over time. A simple version of our approach covers any trajectory of cohort postponement, with the important caveat that the same shift applies to all ages within a cohort. However, the model can readily be extended to include both variation in postponement by age within each cohort and also period effects on the level ("quantum") of fertility which introduce variable cohort quantum.

We begin by presenting the cohort shift model in a simple form without any quantum effects. Then, we include variation in the shape of cohort schedules by allowing period quantum to vary. Finally, we introduce a general model of postponement, which allows us to see how the cohort models, and the other models in the literature fit into a common framework.

2. Models of cohort postponement

Denote the fertility rate at age a and time t by $f(a, t)$. The fertility at age a of the cohort born at time c will be denoted with a subscript

$f_c(a)$. Translation from period to cohort is made using the fact that $t = c + a$, and so $f(a, c + a) = f_c(a)$ and $f_{t-a}(a) = f(a, t)$.

2.1. **The cohort shift model.** In general, shifts in timing could vary by cohort and age, such that

$$f_c(a) = f_0(a - S(a, c)),$$

where f_0 is a standard baseline schedule that would have occurred without postponement and $S(a, c)$ is the "shift", which can vary by cohort and by age. For example, if "40" were the new "30" for the cohort of 1960, then $S(40, 1960) = 10$.

However, a basic model of cohort postponement, which is still quite flexible in that it allows each cohort to postpone by a different amount, assumes uniform age-shifts:

$$f_c(a) = f_0(a - S(c)),$$

By definition the cohort total fertility rate (CTFR) is

$$CTFR_c = \int f_c(a)da.$$

where the unspecified limits of integration span all possible ages (this convention will be used throughout). Shifts within a cohort should not change the CTFR, and indeed we can see that this is the case. Replacing $f_c(a)$ with $f_0(a - S(c))$ gives us $CTFR_c = \int f_0(a - S(c))da$. We evaluate this integral using the change of variables $w = a - S(c)$ to get

$$CTFR_c = \int f_0(w)dw = CTFR_0.$$

The period TFR is influenced by the extent of cohort shifts. Writing

$$TFR(t) = \int f(a, t)da = \int f_0(a - S(c + a))da$$

one can see that the sum of period fertility will depend on $S(c + a)$ Intuitively, this is because in the age-schedule in a given period will depend on the history of cohort shifts. The degree to which the period TFR is influenced by cohort shifts will now be seen.

Define a shift-adjusted period Total Fertility Rate, denoted by $TFR^{\dagger}(t)$, as $\int f(a, t)(1 + S'(c))da$, where $S'(c)$ is the derivative of the shift function with respect to cohort. The reason for this definition is made apparent by the following calculation. First observe that replacing c with $t - a$ in the equation $f(a, c + a) = f_0(a - S(c))$ gives

$$f_0(a - S(t - a)) = f(a, t).$$

Thus

$$TFR^{\dagger}(t) = \int f(a,t)(1+S'(c))da = \int f_0(a-S(t-a)(1+S'(t-a))da$$

and by setting w equal to $a-S(t-a)$ we change variables to get

$$TFR^{\dagger}(t) \int f_0(w)dw$$

which is the period total fertility rate of the baseline schedule, $TFR(0)$. Thus shift-adjusted $TFR^{\dagger}(t)$ recaptures the period TFR that would have been observed in the absence of cohort shifts.

The reason that the definition works is because the age-shifts from cohort-to-cohort are recapitulated in the cross-section from age-to-age. Increasing postponement effectively speeds up the clocks of those in a synthetic cohort within a given period. Likewise, slowing postponement means that the synthetic cohort within a given period will have more exposure at a given fertility rate. The neat thing about our adjustment is that rather than inflating or deflating the time spent at each age, we inflate or deflate the rate in a way that exactly compensates for the compression or extension of age introduced by the cohort shifts.

2.2. **The cohort shift model with period quantum.** Including period effects on the level of fertility within the cohort shift model is straightforward. We denote the period effect by $q(t)$ so that

$$f(a,t) = f_{t-a}(a)q(t) = f_0(a-S(t-a))q(t).$$

Note that the period level effect here is invariant by age, and that the cohort shift effect is invariant by period.

Under this model the Cohort Total Fertility Rate depends on the history of period effects. $CTFR_c = \int f_c(a)q(c+a)da = \int f_0(a - S(c))q(c+a)da$.

Using this model we can still define

$$TFR^{\dagger}(t) = \int f(a,t)(1+S'(c))da,$$

however its meaning has changed somewhat. There is no longer a single cohort TFR that is being recovered; instead $TFR^{\dagger}$ representing a hypothetical cohort exposed to constant level effects of magnitude $q(t)$ over its entire reproductive span. In this way, it is bit like the B-F or K-P measures, except that it is measuring the absence of a long history of cohort postponement rather than only the absence of a recent history of period postponement. The adjusted rate can perhaps be thought of as a pure period measure of quantum one obtains after cohort age shifts are taken into account.

2.3. **Some simple examples.**

Estimation and application of the cohort shift model can be done by estimating $S'(c)$ from data. However there are choices for $S(c)$ which give insight into the consequences of the of the cohort shift model. The following examples are useful to illustrate these consequences.

Example 1: Linear shifts

Following Rodriguez ..., let $S'(c)$ be a constant k. In this case, $TFR^\dagger$ is $(1+k)$ times the observed TFR. Since the fertility schedule $f(a,t)$ is $f_0(a-kt)$, period fertility is shifted but its shape is unchanged, and so the BF formula, $TFR^* = TFR/(1-\mu')$ is also applicable. It follows that $TFR/(1-\mu') = TFR(1+k)$ and so

$$k = \frac{\mu'}{1-\mu'},$$

which is the result obtain by Zeng and Land letting $r_* = k$ and $r = \mu'$.

Example 2: Piecewise linear shifts

The above example generalizes naturally to a situation where the cohort shifts are piecewise linear. Suppose $S'(c)$ is the constant k_1 prior to cohort c_1 and the constant k_2 from cohort c_1 on. Then $TFR^\dagger(t)$ is given by

$$(1+k_2)\int_0^{t-c_1} f(a,t)da \ + \ (1+k_1)\int_{t-c_1}^{\infty} f(a,t)da.$$

Notice that any given time period has only one mean age $\mu(t)$, so that the Bongaarts Feeney adjustment factor $(1-\mu'(t))$, which works nicely for linear shifts, does not lend itself to a piecewise linear scenario.

Example 3: Polynomial shifts

Now consider the case that $S(c)$ is a polynomial in c. It is particularly interesting to center this polynomial at $-\mu(t)$ so that our cohort of interest is the one currently at the mean age of childbearing. So we can write $S(c) = \sum_0^n b_i(c+\mu(t))^i = \sum_0^n b_i(t-(a-\mu(t))^i$ where the coefficients b_i are constants. If our polynomial is quadratic then

$$TFR^\dagger(t) = \int (1+b_1+2b_2t-2b_2(a-\mu(t)))f(a,t)da = (1+b_1+2b_2t)TFR(t)$$

since the term $2b_2(a-\mu(t))$ integrates to zero.

If the polynomial is cubic then $TFR^\dagger(t)$ is

$$\int (1+b_1+2b_2t-2b_2(a-\mu(t))+3b_3(t^2-2t(a-\mu(t)+(a-\mu(t))^2))f(a,t)da$$

$$= (1 + b_1 + 2b_2 t + 3b_3(t^2 + \sigma^2(t)))TFR(t)$$

where $\sigma^2(t)$ is the variance of the schedule $f(a,t)$ with respect to a.

For polynomials of arbitrary degree we can use an approach that will work for any analytic function. Let $G(a) = S'(t-a)$ and expand G in a Taylor series about $\mu(t)$. Then

$$TFR^{\dagger}(t) = \int (1 + G(a)) f(a,t) da =$$

$$TFR(t)(1 + S'(t-\mu(t)) + \sum_{n=2} (-1)^n \frac{S^{(n+1)}(t-\mu(t))}{n!} \mathfrak{K}(t)_n$$

where $\mathfrak{K}(t)_n$ is the centralized nth moment of the fertility schedule $f(a,t)$. If we expect the higher moments to be small, then we can use the following approximation

$$TFR^{\dagger}(t) \cong (1 + S'(t-\mu(t)) + S'''(t-\mu(t))\sigma^2/2)TFR(t).$$

2.4. **An even more general model of cohort postponement that includes B-F, and K-P.** Here, we present a general model of time and age shifts. The idea behind the general model is to define a function $u(a,t)$ that gives the incremental increase in postponement at age a and time t. This, in combination with knowledge of the initial postponement of each cohort, allows a full description of any shift function $S(a,c)$ on the Lexis surface.[1] We first describe this model and then show how it encompasses the models in the literature to date, notably that of Bongaarts and Feeney, Kohler and Philipov, and the cohort shift models presented above.[2]

Let $u(a,t)$ be the incremental increase in postponement at age a and time t. Define the cumulative postponement for cohort c by

$$S(a,c) = \int_0^a u(x, x+c)dx + S(0,c).$$

where $S(0,c)$ is the "initial postponement" of cohort c. Note that partial derivative $S_a(a,c)$ is $u(a, a+c)$ and $S_c(a,c) = \int_0^a u_t(x, x+c)dx + S_c(0,c)$.

[1]Further generalization could proceed by adding quantum terms, and even further generalization could treat each value $S(a,c)$ as the mean of some random variable, and even further, further generalization could consider population mixing of distinct homogeneous populations.

[2]The way we conceive of postponement is as a kind of time and clock shifts. In order to avoid problems of shifts occurring to births that have already occurred and other logical inconsistencies, it is convenient to first allow time and age to shift as specified by the the $S(a,c)$ function and then simply remap take a pre-assigned surface of births $B(a,c)$ and move them accordingly.

To show how observed period values relate to the baseline cohort schedule, let $F_0(a) = \int_0^a f_0(x)dx$ be the cumulative fertility for the baseline cohort. Following Rodriguez, let $F(a,c) = F_0(a - S(a,c))$ be the cumulative fertility for cohort c at age a. The observed fertility rate $f(a,t)$ is then obtained by differentiating $F_0(a - S(a,c))$ with respect to a and then replacing c with $t-a$ to get $f(a,t) = f_0(a - S(a,t-a))(1 - S_a(a,t-a))$. The definition of $TFR^\dagger$ that recovers the baseline schedule is

$$TFR^\dagger(t) = \int f(a,t)\frac{(1 - S_a(a,t-a) + S_c(a,t-a))}{1 - S_a(a,t-a)}da.$$

Notice that we can rewrite this expression by replacing $f(a,t)$ with $f_0(a - S(a,t-a))(1 - S_a(a,t-a))$ to get

$$\int f_0(a - S(a,t-a)(1 - S_a(a,t-a) + S_c(a,t-a))da.$$

Next we set w equal to $a - S(a,t-a)$ and change variables to get $\int f_0(w)dw$ which once again makes $TFR^\dagger(t)$ equal to the baseline total fertility rate.

We now consider various examples of this more sophisticated model.

Example 1: The model presented in the previous section is the special case that $u(a,t) = 0$ and all postponement is determined by the initial postponement $S(0,c)$.

Example 2: Bongaarts and Feeney consider the situation in which postponement is a function $r(t)$ of time which does not vary with age. This situation can be encompassed within our model by setting $u(a,t) = r'(t)$ and $S(0,c) = r(c)$. Then $S_c(a,t-a)$ will be $r'(t)$ and we get

$$TFR^\dagger(t) = \int \frac{f(a,t)}{1 - r'(t)}da = \frac{TFR(t)}{1 - r'(t)}$$

which is exactly the result obtained by Bongaarts and Feeney.

Example 3: In order to investigate the consequences of variance effects in the BF formula, Philipov and Kohler consider a scenario in which cumulative postponement varies linearly with a. They choose a function of the form $S(a,t-a) = a - \bar{a}_0 - (a - \bar{a}_0 - \gamma t)e^{-\delta t}$ where γ and δ are constants and $\bar{a}_0$ is the mean of the baseline schedule f_0.

Using Philipov and Kohler's form for $S(a,t-a)$ we find $S_a(a,t-a) = 1 + e^{-\delta t}(-1 + a\delta - a_0\delta + \gamma - \delta\gamma t)$, $S(0,c) = a_0e^{-\delta c} - a_0 + \gamma c e^{-\delta c}$ and

$S_c(a,c) = e^{-\delta t}(a\delta - a_0\delta + \gamma - \delta\gamma t)$. Thus we calculate $TFR^{\dagger}(t)$ as

$$\int \frac{f(a,t)}{1-\gamma-\delta(a-\bar{a}_0-\gamma t)} da.$$

If $\delta = 0$ then this is $(1-\gamma)^{-1}TFR(t)$. Otherwise, notice that since $\bar{a}_0$ is the mean of the baseline schedule, it follows that

$$0 = \int (w - a_0) f_0(w) dw = \int \frac{e^{-\delta t}(a-\bar{a}_0-\gamma t)}{1-\gamma-\delta(a-\bar{a}_0-\gamma t)} f(a,t) da.$$

Since $\delta \neq 0$ then we can factor $(1-\gamma)e^{-\delta t}\delta^{-1}$ out of the expression on the right to see that

$$(\star) \qquad \int \frac{a-\bar{a}_0-\gamma t}{(1-\gamma)(1-\gamma-\delta(a-\bar{a}_0-\gamma t))} f(a) da = 0.$$

Now via algebra we can write

$$TFR^{\dagger}(t) = \int \frac{f(a,t)}{1-\gamma-\delta(a-\bar{a}_0-\gamma t)} da =$$

$$\int \left[\frac{1}{1-\gamma} + \frac{\delta(a-\bar{a}_0-\gamma t)}{(1-\gamma)(1-\gamma-\delta(a-\bar{a}_0-\gamma t))}\right] f(a,t) da.$$

which using equation $\star$ above is

$$\int \frac{f(a,t)}{1-\gamma} da = \frac{1}{1-\gamma} TFR(t)$$

and thus regardless of the value of δ we recover the Philipov and Kohler formula.

3. Artifacts produced by period tempo-adjustment of cohort postponed fertility

Here we investigate the behavior of the B-F adjustment procedure when applied to cohort postponement (Note: one could also consider the place in which adjustment age-invariant cohort postponement is applied to a period postponed world).

[This important section needs to be written.]

Based on simulations, not reported here, we see that the two estimates differ at periods at the onset and conclusion of cohort shift transitions. Our suspicion is that the error is a function of the third-derivative of S(c), which is related to the rate of change in the variance in period fertility distributions, found to be the key factor by Kohler and Philipov.

4. Estimation of $TFR^{\dagger}$ with real data

To estimate the shift-adjusted TFR, we need separate estimates of a baseline schedule $f_0(a)$, the shifts by cohort $S(c)$, and the period quantum $q(t)$.

Using iterative calculation, all three of these quantities can at least in theory be estimated by adding some constraints, e.g., that q(t) averages out to 1.0. Initial estimates of the baseline schedule can be obtained by averaging out the observed cohort fertility schedules at all ages, including those cohorts that are truncated. Initial estimates of the q(t) parameter can be obtained from the period TFR, appropriately normalized.

[This section still needs to be developed and written. Our suspicion is that the estimates of TFR$^d agger$ will not be dramatically different than B-F's estimates but that the cohort model will be superior in some interesting situations, e.g., Eastern Europe.]

5. Discussion

The cohort shift model is at least an alternative formulation to the period perspective of Bongaarts and Feeney. At best, it will prove to be estimable and in some cases provide superior estimates of the underlying level of fertility.

The cohort-shift model fits into a larger class of shift models, which include all of the shift models we know of the literature to date. This general form for shift models does not – at least to us – present a full set of tractable analytic forms for the postponement function $u(a, t)$. However, it does, at least, allow us to understand how all of the different models to date relate to one another. In the future, it may also lead to useful flexible formulations of $u(a, t)$ that are both analytically understandable and behaviorally defensible.